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Problems with different time scales

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CONTENTS

1. An example

In this section we discuss a very simple problem. Consider the scalar initial value problem

$$\begin{aligned} \varepsilon y' &= ay + \mathrm{e}^{\mathrm{i}t}, \quad t \geq 0, \\ y(0) &= y_0. \end{aligned} \tag{1.1}$$

Here $\varepsilon > 0$ is a small constant and $a = a_1 + \mathrm{i}a_2$, a_1, a_2 real, is a complex number with $|a| = 1$. We can write down the solution of (1.1) explicitly. It is

$$y = y^{\mathrm{S}} + y^{\mathrm{F}},$$

where

$$y^{\mathrm{S}} = -\frac{1}{a - \mathrm{i}\varepsilon}\mathrm{e}^{\mathrm{i}t}$$

is the forced solution and

$$y^{\mathrm{F}} = \left(y_0 + \frac{1}{a - \mathrm{i}\varepsilon}\right)\mathrm{e}^{a/\varepsilon\, t}$$

is a solution of the homogeneous equation

$$\varepsilon v' = av.$$

y^S varies on the time scale '1' while y^F varies on the much faster scale $1/\varepsilon$. We say that y^S, y^F vary on the slow and fast scale, respectively. We use also the phrase: y^S and y^F are the slow and the fast part of the solution, respectively.

There are three different possibilities.

Case 1. $a_1 \gg \varepsilon$. In this case y^F grows rapidly and dominates the solution. We are not interested in this case.

Case 2. $a_1 \gg -\varepsilon$. Now y^F decays rapidly. Therefore, outside a boundary layer, the solution of (1.1) is essentially represented by y^S.

Case 3. $a_1 = 0$. For general initial data both scales are present for all times. However, if

$$y_0 + \frac{1}{a - i\varepsilon} = 0,$$

then $y^F = 0$ and the solution varies on the slow scale only.

In this survey article we shall mainly discuss the third case.

In applications like meteorology, oceanography and plasma physics one is often only interested in solutions, which vary on the slow scale. However, the data are such that the fast scale is present anyway. Therefore we shall develop a theory, which leads to a systematic way to 'initialize' the data such that the fast scale is not excited. This theory is based on a very simple principle. If $y(t)$ varies on the slow time scale, then

$$d^\nu y(t)/dt^\nu \approx \mathcal{O}(1), \quad \nu = 0, 1, 2, \ldots, p,$$

where $p > 1$ is a suitable number. Therefore our principle as follows.

Choose the initial data $y(0) = y_0$ such that at $t = 0$

$$d^\nu y(0)/dt^\nu \approx \mathcal{O}(1), \quad \nu = 0, 1, 2, \ldots, p. \tag{1.2}$$

We shall call this procedure *the bounded derivative principle.*

Let us apply the principle to our example. We think of ε as a small parameter, which approaches 0, and we want to choose the initial data such that the derivatives at $t = 0$ are bounded independently of ε.

$dy(0)/dt$ is bounded independently of ε, if and only if

$$ay(0) + 1 = \mathcal{O}(\varepsilon), \quad \text{i.e.} \quad y(0) = -1/a + \mathcal{O}(\varepsilon).$$

Thus the initial data are determined up to terms of order $\mathcal{O}(\varepsilon)$.

For the second derivative we have

$$\varepsilon^2 y'' = \varepsilon a y' + \varepsilon e^{it} = a^2 y + a e^{it} + i\varepsilon e^{it}.$$

Therefore $d^2y(0)/dt^2$ is bounded independently of ε, if and only if

$$y(0) = -1/a - i\varepsilon/a^2 + \mathcal{O}(\varepsilon^2).$$

Thus the initial data are determined up to terms of order $\mathcal{O}(\varepsilon^2)$.

An easy calculation shows that the initial data are determined up to terms of order $\mathcal{O}(\varepsilon^p)$, if and only if the first p derivatives are bounded indepentently of ε.

Earlier we have shown that $y^F \equiv 0$, if

$$y(0) = -\frac{1}{a - i\varepsilon} = -\frac{1}{a}\sum_{\nu=0}^{\infty}\left(\frac{i\varepsilon}{a}\right)^{\nu}.$$

The bounded derivative principle gives us the first p terms in the power series expansion.

We want to prove that for general nonlinear systems the bounded derivative principle lets us determine the slow solution to any order.

The bounded derivative principle is very much connected with asymptotic expanstions. To discuss the connection we consider the slightly more general equation

$$\begin{aligned} \varepsilon y' &= ia(t)y + f(t), \\ y(0) &= y_0, \end{aligned} \tag{1.3}$$

where $a(t), f(t) \in C^\infty(t)$ and $a(t) \geq a_0$, $a_0 =$ constant > 0.

First we shall show that the bounded derivative principle is valid. The construction will be generalized to systems in the next section.

If $y(t)$ is a slow solution, then

$$y(t) \approx \varphi_0(t) =: i\,f(t)/a(t).$$

This suggests the substitution

$$y(t) = \varphi_0(t) + y_1(t). \tag{1.4}$$

Introducing (1.4) into (1.3) gives us

$$\begin{aligned} \varepsilon y_1' &= ia(t)y_1 + \varepsilon f_1(t), \quad f_1(t) =: -\varphi_0'(t), \\ y_1(0) &= y_{10} =: y_0 - \varphi_0(0). \end{aligned} \tag{1.5}$$

(1.5) is of the same form as (1.3). However, the forcing is reduced to order $\mathcal{O}(\varepsilon)$. We can repeat the process. After p steps we obtain

$$y(t) = \psi_{p-1}(t) + y_p(t), \quad \psi_{p-1}(t) = \sum_{j=0}^{p-1}\varepsilon^j\varphi_j(t), \tag{1.6}$$

where $y_p(t)$ solves

$$\begin{aligned} \varepsilon y_p' &= ia(t)y_p + \varepsilon^p f_p(t), \\ y_p(0) &= y_0 - \psi_{p-1}(0). \end{aligned} \tag{1.7}$$

The solution of (1.7) can be written as

$$y_p = \overline{y} + \overline{\overline{y}},$$

where $\overline{y}$ is the forced solution satisfying

$$\begin{aligned} \varepsilon \overline{y}' &= \mathrm{i}a(t)\overline{y} + \varepsilon^p f_p(t), \\ \overline{y}(0) &= 0, \end{aligned} \tag{1.8}$$

and $\overline{\overline{y}}$ solves

$$\begin{aligned} \varepsilon \overline{\overline{y}}' &= \mathrm{i}a(t)\overline{\overline{y}}, \\ \overline{\overline{y}}(0) &= \overline{\overline{y}}_{p0}. \end{aligned} \tag{1.9}$$

By Duhamel's principle

$$|\mathrm{d}^j \overline{y}/\mathrm{d}t^j| \leq \text{ constant } \times \varepsilon^{p-j-1}$$

in any finite time interval $0 \leq t \leq T$. Thus $\overline{y}$ has $p-1$ derivatives bounded independently of ε.

Now apply the bounded derivative principle to (1.9). An easy calculation shows that $\overline{\overline{y}}(t)$ and therefore also $y(t)$ have $p-1$ derivatives bounded independently of ε at $t=0$ if and only if

$$\overline{\overline{y}}(0) = \mathcal{O}(\varepsilon^{p-1}), \quad \text{i.e.} \quad y(0) = \psi_{p-1}(0) + \mathcal{O}(\varepsilon^{p-1}). \tag{1.10}$$

If (1.10) holds, then $\overline{\overline{y}}(t)$ and therefore also

$$y(t) = \psi_{p-1}(t) + \overline{y}(t) + \overline{\overline{y}}(t) = \psi_{p-1}(t) + \mathcal{O}(\varepsilon^{p-1}), \tag{1.11}$$

have $p-1$ derivatives bounded independently of ε in any finite time interval. This shows that the bounded derivative principle is valid.

(1.10) and (1.11) also show that equation (1.3) has essentially a unique slow solution and that $\psi_{p-1}(t)$ represents the first p terms of its asymptotic expansion. One can determine the initial data either by the bounded derivative principle or by calculating the asymptotic expansion in a neighbourhood of $t=0$ and use $\psi_{p-1}(0)$ as initial data.

We have calculated the asymptotic expansion by substitution. Instead we can also determine it by the iteration

$$\varepsilon(y^{(n-1)})' = \mathrm{i}a y^{(n)} + f, \quad y^{(-1)} \equiv 0, \ n = 0, 1, 2, \ldots .$$

An easy calculation shows that

$$y^{(p)} = \psi_p.$$

Our construction depends heavily on the assumption that $a(t), f(t)$ have derivatives of order $\mathcal{O}(1)$ and that $a(t) \geq a_0 > 0$. If, for example,

$$f(t) = \begin{cases} 0 & \text{for } 0 \leq t < \frac{1}{2} \\ 1 & \text{for } t \geq \frac{1}{2}, \end{cases}$$

then the asymptotic expansion tells us that the slow solution is given by

$$y(t) = \begin{cases} 0 & \text{for } 0 \le t < \frac{1}{2} \\ \frac{2i}{a(t)} + \mathcal{O}(\varepsilon) & \text{for } t \ge \frac{1}{2}. \end{cases}$$

Thus the solution of (1.3) with initial data $y(0) = 0$ will become highly oscillatory for $t \ge \frac{1}{2}$. Correspondingly, if

$$a(t) = (t - t_0)a_1(t),$$

then a solution, which is slow for $t < t_0$, becomes in general highly oscillatory for $t > t_0$.

2. Systems of ordinary differential equations

2.1. Form of the systems and assumptions

In applications the systems are real and often have the form

$$w_t = \frac{1}{\varepsilon} A_1(t)w + f_1(w,t), \quad 0 < \varepsilon \le \varepsilon_0, \tag{2.1}$$

i.e. the large part of the right-hand side is a linear function of w. We assume that $A_1(t)$ has constant rank, i.e. there is a smooth transformation $S(t)$ such that

$$S^{-1}(t)A_1(t)S(t) = \begin{pmatrix} A(t) & 0 \\ 0 & 0 \end{pmatrix}, \quad \det A \neq 0.$$

Changing the dependent variables accordingly, we obtain a system of the form

$$\begin{aligned} \varepsilon y' &= (A(t) + \varepsilon C(v,y,t))y + f(v,t), \quad 0 < \varepsilon \le \varepsilon_0, \\ v' &= g(v,y,t), \end{aligned} \tag{2.2}$$

where

$$y(t), f(v,t) \in \mathbb{R}^m, \; v(t), g(v,y,t) \in \mathbb{R}^n, \; A(t), C(v,y,t) \in \mathbb{R}^{m \times m}.$$

We want to show that the results of the previous section can be generalized to systems (2.2). We follow here closely the presentation in Kreiss and Lorenz (1991). (See also Kreiss (1979) and Sacker (1965).)

To be precise, we shall use the following terminology

Definition 2.1 Let $w(t,\varepsilon)$ denote a function defined for $0 \le t \le T$, $0 < \varepsilon \le \varepsilon_0$. We say that it is slow to order p in $0 \le t \le T$ if $w \in C^p(0,T)$ and if

$$\sup_{0<\varepsilon\le\varepsilon_0} \max_{0\le t\le T} |\partial^j w/\partial t^j| < \infty, \quad j = 0,1,\dots,p. \tag{2.3}$$

We say that w is slow if (2.3) holds for any p.

Our main assumption is

Assumption 2.1
(i) For all $t \geq 0$

$$A(t) + A^*(t) \leq 0, \quad \det A(t) \neq 0.$$

(ii)

$$A(t), A^{-1}(t), C(v, y, t), f(v, t), g(v, y, t)$$

are C^∞-functions of their arguments with bounded derivatives. They may also depend on ε but we assume that the bounds are uniform in ε.

2.2. The bounded derivative principle and asymptotic expansions

We shall separate the fast and slow variables by a suitable substitution. If $y(t), v(t)$ is a slow solution, then $\varepsilon y'(t) = \mathcal{O}(\varepsilon)$ and to the first approximation

$$y(t) = -A^{-1}(t) f(v, t).$$

This suggests the substitution

$$y(t) = \Phi_0(v, t) + y_1(t), \quad \Phi_0(v, t) = -A^{-1}(t) f(v, t).$$

Introducing (2.3) into (2.2) gives us

$$\begin{aligned} \varepsilon y_1'(t) + \varepsilon(\partial \Phi_0(v, t)/\partial v) g(v, \Phi_0 + y_1, t) + \varepsilon \Phi_{0t}(v, t) \\ = A(t) y_1(t) + \varepsilon C(v, \Phi_0 + y_1, t)(\Phi_0(v, t) + y_1(t)) \\ v'(t) = g(v, \Phi_0 + y_1, t), \end{aligned}$$

i.e.

$$\begin{aligned} \varepsilon y_1' &= (A(t) + \varepsilon C_1(v, y_1, t)) y_1 + \varepsilon f_1(v, t) \\ v' &= g_1(v, y_1, t), \end{aligned} \tag{2.4}$$

where

$$f_1(v, t) = -(\partial \Phi_0(v, t)/\partial v) g(v, \Phi_0, t) - \Phi_{0t}(v, t).$$

Thus $y_1(t)$ satisfies a differential equation of the same form as $y(t)$ but the forcing is reduced to order $\mathcal{O}(\varepsilon)$.

We can repeat the process and obtain

Theorem 2.1 One can construct slow functions $\Phi_0(v, t), \Phi_1(v, t), \ldots$ with the following properties. If one substitutes

$$\begin{aligned} y(t) &= \psi_{p-1}(v, t) + y_p(t), \\ \psi(v, t) &= \Phi_0(v, t) + \varepsilon \Phi_1(v, t) + \cdots + \varepsilon^{p-1} \Phi_{p-1}(v, t), \end{aligned}$$

into (2.2), then $y_p(t), v(t)$ satisfy a system

$$\begin{aligned} \varepsilon y_p' &= (A(t) + \varepsilon C_p(v, y_p, t)) y_p + \varepsilon^p f_p(v, t) \\ v' &= g_p(v, y_p, t), \end{aligned} \tag{2.5}$$

where C_p, f_p, g_p have uniformly bounded derivatives with respect to all variables. Differentiating (2.5) we obtain immediately

Theorem 2.2 $y_p(t), v(t)$ and therefore also $y(t), v(t)$ are slow to order p in any fixed time interval $0 \le t \le T$, if and only if

$$y_p(0) = \mathcal{O}(\varepsilon^p), \quad \text{i.e.} \quad \mathrm{d}^j y_p(0)/\mathrm{d}t^j = \mathcal{O}(\varepsilon^{p-j}), \quad j = 0, 1, \ldots, p. \tag{2.6a}$$

(2.6a) holds if and only if for the original variables

$$\mathrm{d}^j y(0)/\mathrm{d}t^j = \mathcal{O}(1) \quad \text{for} \quad j = 0, 1, \ldots, p. \tag{2.6b}$$

Thus the bounded derivative principle is valid.

As in the previous section we can use the last theorem to initialize the data, i.e. for a given $v_0 = v(0)$ find $y_0 = y(0)$ such that the solution is slow to order p. We find the relations by enforcing (2.6b). Essentially we have to calculate $\psi_{p-1}(v(0), 0)$. The process can become quite complicated. Therefore it is often easier to determine the relations by iteration. We have

Theorem 2.3 Let $y^{(0)}(t) \equiv 0$. p iterations of

$$\begin{aligned} \varepsilon(y^{(n-1)})' &= (A(t) + \varepsilon C_p(v^{(n)}, y^{(n)}, t)y^{(n)} + f(v^{(n)}, t), \\ (v^{(n)})' &= g(v^{(n)}, y^{(n)}, t), \quad v^{(n)}(0) = v_0, \ n = 0, 1, \ldots, p-1, \end{aligned}$$

determines a solution of (2.2), which is slow to order p and which, for a given v_0, is unique up to terms of order $\mathcal{O}(\varepsilon^p)$.

We can solve this iteration numerically in a neighbourhood of $t = 0$ and use the resulting $y^{(p-1)}(0)$ as initial data to solve the system (2.2) in large time intervals.

Under the following additional assumptions the estimates can be extended to all times.

Assumption 2.2 There is a constant $\beta > 0$ and an integer $q_0 \ge 1$ such that

$$C_{q_0}(v, y, t) + C^*_{q_0}(v, y, t) \le -\beta\varepsilon^{q_0 - 1} I$$

for all v, y, ε.

(For the proof see Kreiss and Lorenz (1991).)

2.3. Existence of a slow manifold

Theorem 2.2 tells us that we can choose $y(0)$ as a function of $v(0)$ such that the resulting solution is slow to order p. In general the relationship between y and v depends on p. If we want the solution to be slow to order $p+1$, then we have to change the relation by terms of order ε^p.

For practical purposes this result is completely satisfactory. However, an

interesting mathematical question is: Can we determine $y(0)$ as a function of $v(0)$ such that the resulting solution is slow to any order?

There is one trivial case, where this is so. If $f(v,t)$ and all its partial derivatives with respect to v and t vanish at $t=0$, then the initial data

$$y(0)=0,\ v(0)=v_0$$

guarantee solutions, which are slow to any order. Also, for given v_0 the slow solution is unique to any order in ε.

In Kreiss and Lorenz (1991) we have reduced the general case to the above by constructing a substitution

$$y=\Phi(v,t,\varepsilon)+\tilde{y},\quad 0\le t\le T, \tag{2.7}$$

such that $f(v,t)\equiv 0$. We have also given conditions such that the substitution exists for all times. (See also Sacker (1965), Kopell (1985), Fenichel (1985).)

(2.7) shows that the slow solution forms a manifold represented by

$$\begin{aligned} y &= \Phi(v,t,\varepsilon)\\ v' &= g(v,y,t). \end{aligned} \tag{2.8}$$

2.4. Interaction between the fast and the slow scale

Consider the system (2.2) and choose the initial data by

$$v(0)=v_0,\quad y(0)=\Phi(v_0,0,\varepsilon)$$

to obtain the slow solution $v^{\mathrm{S}}(t), y^{\mathrm{S}}(t)$. Now perturb $y(0)$ and consider

$$v(0)=v_0,\quad y(0)=\Phi(v_0,0,\varepsilon)+\delta$$

and denote the resulting solution by $v_\delta(t), y_\delta(t)$. In general the fast scale is excited and $y^{\mathrm{S}}(t)$ will be of order $\mathcal{O}(|\delta|)$. We want to show that the effect on the slow part of the solution is much smaller than $|\delta|$.

Theorem 2.4 Let $0\le t\le T$ be a fixed time interval. For sufficiently small $\varepsilon, |\delta|$ there is a constant c_0 such that

$$|v^{\mathrm{S}}(t)-v_\delta(t)|\le c_0(\varepsilon|\delta|+|\delta|^2).$$

Proof. We shall only indicate the proof. For more details see Kreiss and Lorenz (1991). Without restriction we can assume that the system (2.2) has the form

$$\begin{aligned} \varepsilon y' &= (A(t)+\varepsilon C(v,y,t))y\\ v' &= g(v,y,t). \end{aligned} \tag{2.9}$$

Otherwise we perform the substitution (2.7). The initial data for $v^{\mathrm{S}}(t), y^{\mathrm{S}}(t)$ are

$$v^{\mathrm{S}}(0)=v_0,\quad y^{\mathrm{S}}(0)=0,\quad \text{i.e.}\quad y^{\mathrm{S}}(t)\equiv 0,$$

and for $v_\delta(t), y_\delta(t)$

$$v_\delta(0) = v_0, \quad y_\delta(t) = \delta.$$

To first approximation $y_\delta(t)$ satisfies

$$\varepsilon y_\delta'(t) = (A(t) + \varepsilon C(v^S, 0, t))y_\delta, \quad \text{i.e.} \quad |y_\delta| = \mathcal{O}(\delta),$$

and $w = v^S - v_\delta$ solves

$$\begin{aligned} w' &= (\partial g(v^S, 0, t)/\partial v)w + \partial g(v^S, 0, t)/\partial y y_\delta(t), \\ w(0) &= 0. \end{aligned} \tag{2.10}$$

Therefore

$$\begin{aligned} w(t) &= \int_0^t S(t, \xi)(\partial g(v^S, 0, \xi)/\partial y) y_\delta(\xi)\, \mathrm{d}\xi \\ &= \varepsilon \int_0^t S(t, \xi)(\partial g(v^S, 0, \xi)/\partial y)(A + \varepsilon C)^{-1} y_\delta'(\xi)\, \mathrm{d}\xi. \end{aligned}$$

Here $S(t, \xi)$ is the solution operator of the homogeneous equation

$$u' = (\partial g(v^S, 0, t)/\partial v)u.$$

Integration by parts shows that the last integral is of order $\varepsilon\delta$. Also, we have neglected only terms of order $\mathcal{O}(\varepsilon\delta) + \mathcal{O}(\delta^2)$ and therefore the theorem follows. □

The last theorem is important in applications. Often one is only interested in slow solutions. In these cases one has to prepare the initial data in such a way that the fast scale is not excited. Practically one can never remove the fast scale completely. The theorem says that the effect of the fast scale on the slow scale can be neglected, provided a moderate amount of data preparation has been performed. Also, the fast scale can be removed by post-filtering.

3. Numerical methods for ordinary differential equations

3.1. An example

We consider equation (1.3)

$$\begin{aligned} \varepsilon y' &= \mathrm{i}\, a(t) y + f(t) \\ y(0) &= y_0. \end{aligned} \tag{3.1}$$

To begin with we want only to calculate the slow solution. There are a number of possibilities.

Asymptotic expansion, i.e.

$$y^S(t) = \frac{\mathrm{i} f(t)}{a(t)} + \frac{\varepsilon}{a(t)} \left(\frac{f(t)}{a(t)} \right)' + \cdots + \mathcal{O}(\varepsilon^2). \tag{3.2}$$

Difference approximation. If we are willing to use a time step $k \ll \varepsilon$, then any of the standard explicit techniques can be used. However, we want only to calculate the slow solution and therefore we only want to resolve the slow scale, i.e. we want to use a time step k with $\varepsilon \ll k \ll 1$. Therefore we have to use an implicit method. It has to be stable on the imaginary axis and therefore the order of accuracy of a stable multi-step method is restricted to one or two. We shall discuss the *implicit Euler scheme* and *the midpoint rule.*

Let $k > 0$ denote the time step, $t_j = jk$, $j = 0, 1, \ldots$, the grid and denote by $u_j = u(jk)$ the values of u on the grid. Then the *implicit Euler* scheme has the form

$$\varepsilon(u_{n+1} - u_n) = k(\mathrm{i}a_{n+1}u_{n+1} + f_{n+1}). \tag{3.3}$$

As for the continuous case we can derive an asymptotic expansion. Let

$$u_n = \frac{\mathrm{i}f_n}{a_n} + \tilde{u}_n.$$

Then $\tilde{u}_n$ is the solution of

$$\varepsilon(\tilde{u}_{n+1} - \tilde{u}_n) = k\left(\mathrm{i}a_{n+1}\tilde{u}_{n+1} - \mathrm{i}D_-\left(\frac{f_{n+1}}{a_{n+1}}\right)\right), \quad D_-g_{n+1} =: \frac{g_{n+1} - g_n}{k}.$$

Repeating the process we obtain an asymptotic expansion of the slow discrete solution

$$u_n^{\mathrm{S}} = \frac{\mathrm{i}f_n}{a_n} + \frac{\varepsilon}{a_n}D_-\left(\frac{f_n}{a_n}\right) + \mathcal{O}(\varepsilon^2). \tag{3.4}$$

Thus

$$\begin{aligned} |y^{\mathrm{S}}(t_n) - u_n^{\mathrm{S}}| &= \frac{\varepsilon}{|a_n|}\left|\left(\frac{f(t_n)}{a(t_n)}\right)' - D_-\left(\frac{f_n}{a_n}\right)\right| + \mathcal{O}(\varepsilon k^2) \\ &= \mathcal{O}(\varepsilon k). \end{aligned}$$

If we have chosen the initial data by

$$y_0 = y^{\mathrm{S}}(0), \tag{3.5}$$

then the fast part of the discrete solution satisfies

$$\begin{aligned} v_{n+1} &= \kappa v_n, \quad \kappa = \frac{1}{1 - \mathrm{i}(k/\varepsilon)}, \\ v_0 &= y_0 - u^{\mathrm{S}}(0) = \mathcal{O}(\varepsilon k). \end{aligned}$$

Thus the fast part of the discrete solution is, in general, not zero. However, $|\kappa| \ll$ and therefore

$$v_n = \kappa^n v_0 \tag{3.6}$$

shows that v_n converges rapidly to zero regardless of how we choose the

initial data u_0. Thus the implicit Euler method will always determine the slow solution, if $k \gg \varepsilon$.

The midpoint rule is given by

$$\varepsilon(u_{n+1} - u_n) = \frac{k}{2}(\mathrm{i}\, a_{n+1}u_{n+1} + f_{n+1} + \mathrm{i}\, a_n u_n + f_n). \tag{3.7}$$

Now the discrete slow solution satisfies

$$u_n^{\mathrm{S}} = y^{\mathrm{S}}(t_n) + \mathcal{O}(\varepsilon k^2). \tag{3.8}$$

The fast part is the solution of

$$\begin{aligned} v_{n+1} &= \kappa v_n, \quad \kappa = \frac{1 + \frac{1}{2}\mathrm{i}k/\varepsilon}{1 - \frac{1}{2}\mathrm{i}k/\varepsilon} = -\exp\left(\frac{-4\mathrm{i}\varepsilon}{k}\right) + \mathcal{O}\left(\frac{\varepsilon^3}{k^3}\right) \approx -1, \\ v_0 &= u_0 - u_0^{\mathrm{S}}. \end{aligned} \tag{3.9}$$

In this case the fast part will not be damped. Instead it will oscillate like a ± 1 wave. If we choose $u_0 = y^{\mathrm{S}}(0)$, then $v_0 = \mathcal{O}(\varepsilon k^2)$ is small.

The following local smoothing procedure can be used to decrease the amplitude of the fast wave, even if v_0 is large. (See Lindberg (1971).)

1 Starting with u_0 calculate u_1, u_2.

2 Determine new initial data at $t = k$ by

$$\tilde{u}_1 = u_1 + \tfrac{1}{4}(u_2 - 2u_1 + u_0).$$

Repeat the process starting at $t = k$. We have

$$u_j = u_j^{\mathrm{S}} + \kappa^j(u_0 - u_0^{\mathrm{S}}).$$

Therefore

$$\begin{aligned} \tilde{u}_1 &= u_1^{\mathrm{S}} + \tfrac{1}{4}(u_2^{\mathrm{S}} - 2u_1^{\mathrm{S}} + u_0^{\mathrm{S}}) + (\kappa + \tfrac{1}{4}(\kappa - 1)^2)(u_0 - u_0^{\mathrm{S}}) \\ &= u_1^{\mathrm{S}} + \mathcal{O}(k^2) + \mathcal{O}(\varepsilon/k)(u_0 - u_0^{\mathrm{S}}). \end{aligned}$$

Thus the amplitude of the fast solution has been reduced by a factor $\mathcal{O}(\varepsilon/k)$. Generalizations are treated in Majda (1983).

Richardson extrapolation. As we have said in the beginning: If ε is very small, then it is uneconomical to use an explicit method. However, in applications the systems can be very large and it can therefore be quite expensive to solve the linear systems connected with the implicit methods. We know that the slow solution can be expanded into an asymptotic series in ε.

$$u_n^{\mathrm{S}}(k) = \Phi_0 + \varepsilon\Phi_1 + \mathcal{O}(\varepsilon^2).$$

Therefore we can use *Richardson extrapolation.* We change ε to a more moderate value $\varepsilon^* \gg \varepsilon$ and calculate

$$\begin{aligned} u_n^S(\varepsilon^*) &= \Phi_0 + \varepsilon^* \Phi_1 + \mathcal{O}(\varepsilon^{*2}) \\ u_n^S(2\varepsilon^*) &= \Phi_0 + 2\varepsilon^* \Phi_1 + \mathcal{O}(\varepsilon^{*2}) \end{aligned}$$

Then

$$\begin{aligned} \varepsilon^* \Phi_1 &= u_n^S(2\varepsilon^*) - u_n^S(\varepsilon^*) + \mathcal{O}(\varepsilon^{*2}) \\ \Phi_0 &= u_n^S(\varepsilon^*) - (u_n^S(2\varepsilon^*) - u_n^S(\varepsilon)) + \mathcal{O}(\varepsilon^{*2}), \end{aligned}$$

i.e.

$$u_n^S(\varepsilon) = 2u_n^S(\varepsilon^*) - u_n^S(2\varepsilon^*) + \frac{\varepsilon}{\varepsilon^*}(u_n^S(2\varepsilon^*) - u_n^S(\varepsilon^*)) + \mathcal{O}(\varepsilon^{*2}).$$

For moderate values of ε^* we may be able to use an explicit method. The main difficulty is that the initial data have to properly initialized, because Richardson extrapolation does not work for the fast part of the solution.

Until now we have concentrated on calculating the slow solution. If we also want to calculate the fast part of the solution, then we have two possibilities.

1 Use a difference method and resolve the fast scale, i.e. choose $k \ll \varepsilon$.

2 Solve the homogeneous equation

$$\begin{aligned} \varepsilon v' &= \mathrm{i} a(t) v, \\ v(0) &= v_0, \end{aligned}$$

analytically:

$$v(t) = \exp\left[\left(\frac{\mathrm{i}}{\varepsilon}\right) \int_0^t a(\xi)\, \mathrm{d}\xi\right] v(0).$$

3.2. Slow solutions of fast systems

We consider systems

$$\begin{aligned} \varepsilon y' &= A(t) y + f(t) \\ y(0) &= y_0, \end{aligned} \tag{3.10}$$

where $A(t), f(t)$ satisfy Assumption 1.1. We can calculate the slow solution in the same way as for Example 3.1.

Asymptotic expansion.

$$y^S(t) = -A^{-1}(t) f(t) + \varepsilon A^{-1}(t)(A^{-1}(t) f(t))' + \mathcal{O}(\varepsilon^2).$$

Implicit difference approximation. The implicit Euler scheme has the form

$$\varepsilon(u_{n+1} - u_n) = k(A_{n+1}u_{n+1} + f_{n+1}).$$

For the slow part of its solution we have the asymptotic expansion

$$u_n^S = -A_n^{-1}f_n + \varepsilon A_n^{-1}D_-(A_n^{-1}f_n) + \mathcal{O}(\varepsilon^2),$$

and the fast part satisfies

$$v_{n+1} = \left(I - \frac{k}{\varepsilon}A_{n+1}\right)^{-1} v_n = -\frac{\varepsilon}{k}A_{n+1}^{-1}\left(I - \frac{\varepsilon}{k}A_{n+1}^{-1}\right)v_n.$$

Thus, if $\varepsilon \ll k$, then

$$\left|\frac{\varepsilon}{k}A_{n+1}^{-1}\right| \ll 1$$

and v_n converges rapidly to zero, i.e. the implicit Euler scheme gives us the slow solution regardless of the initial data.

The midpoint rule. The same arguments as in the scalar case show that the slow part of the solution can be described by an asymptotic expansion and that the fast part becomes a ± 1-wave, which can be damped by a local smoothing.

Richardson extrapolation. The possibility of Richardson extrapolation depends only on the existence of asymptotic expansions. Therefore we can also use it here.

3.3. Slow solution of the full system

We consider the system (2.2). All the methods discussed in the previous section can be generalized.

Asymptotic expansions. By Theorem 2.3 the first term $(v^{(0)}, y^{(0)})$ is the solution of

$$\begin{aligned} 0 &= A(t)y^{(0)} + f(v^{(0)}, t) \\ (v^{(0)})' &= g(v^{(0)}, y^{(0}, t) \\ v^{(0)}(0) &= v_0. \end{aligned} \tag{3.11}$$

Higher order terms are obtained by the iteration

$$\begin{aligned} \varepsilon(y^{(j)})' &= A(t)y^{(j+1)} + \varepsilon C(v, y^{(j)}, t)y^{(j)} + f(v, t) \\ (v^{(j+1)})' &= g(v^{(j+1)}, y^{(j+1)}, t) \\ v^{(j+1)}(0) &= v_0. \end{aligned} \tag{3.12}$$

The differential equations can be solved by any standard method.

Implicit difference methods. We can use *the backward Euler scheme* or *the midpoint rule* for the complete system. However, for less work we obtain better accuracy, if we apply these schemes to the fast part (y-variables) only. For example, we can use the Euler scheme for the y-variables and a stable *Adam's method* for

$$\begin{aligned} \varepsilon(\tilde{y}_{n+1} - \tilde{y}_n) &= k(A_{n+1} + \varepsilon C_{n+1})\tilde{y}_{n+1} + \tilde{f}_{n+1}), \quad \tilde{f}_{n+1} = f(\tilde{v}_{n+1}, t_{n+1}), \\ \tilde{v}_{n+1} &= \tilde{v}_n + \sum_j \beta_j \tilde{g}_{n-j}, \quad \tilde{g}_{n-j} = g(\tilde{v}_{n-j}, \tilde{y}_{n-j}, t_{n-j}). \end{aligned} \tag{3.13}$$

We can also develop the solutions of (3.13) into an asymptotic expansion and compare it with (3.11) and (3.12). This results in a satisfactory error estimate. However, the stability of the method has not been investigated.

In applications the system has often the form (2.1) and one uses a combination of *leap-frog* and the midpoint rule to solve it, i.e.

$$\tilde{w}_{n+1} - \tilde{w}_{n-1} = \frac{k}{\varepsilon}(A_{1\,n+1}\tilde{w}_{n+1} + A_{1\,n}\tilde{w}_n) + 2k\tilde{f}_{1\,n}. \tag{3.14}$$

We shall give a truncation error and stability analysis of the method.

Let $w(x,t)$ be a slow solution of (2.1) and introduce it into (3.14). We obtain

$$\begin{aligned} & w_{n+1} - w_{n-1} - \frac{k}{\varepsilon}(A_{1\,n+1}w_{n+1} + A_{1\,n}w_n) - 2kf_{1\,n} \\ = \; & w_{n+1} - w_{n-1} - k(w'_{n+1} + w'_{n-1}) + k(f_{1\,n+1} + f_{1\,n-1} - 2f_{1\,n}) \\ & = ck^3 w'''_n + \mathcal{O}(k^4). \end{aligned}$$

Thus the method is second order accurate. To discuss the stability we linearize (3.14) and freeze coefficients, i.e. we consider

$$\tilde{v}_{n+1} - \tilde{v}_{n-1} = \frac{k}{\varepsilon}(A_1\tilde{v}_{n+1} + A_1\tilde{v}_{n-1}) + 2kB\tilde{v}_n, \tag{3.15}$$

where A, B are constant matrices. The stability follows from

Theorem 3.3 Assume that

$$A_1 + A_1^* \le 0, \quad B = -B^*, \quad |kB| \le 1 - \delta,$$

then

$$\delta(|\tilde{v}_{n+1}|^2 + |\tilde{v}_n|^2) \le 2(|\tilde{v}_1|^2 + |\tilde{v}_0|^2). \tag{3.16}$$

Proof. Multiplying (3.15) by $\tilde{v}_{n+1} + \tilde{v}_{n-1}$ gives us

$$\begin{aligned} |\tilde{v}_{n+1}|^2 - |\tilde{v}_{n-1}|^2 &= \frac{k}{\varepsilon}\langle \tilde{v}_{n+1} + \tilde{v}_{n-1}, A_1(\tilde{v}_{n+1} + \tilde{v}_{n-1})\rangle \\ &\quad + 2k\langle \tilde{v}_{n+1}, B\tilde{v}_n\rangle + 2k\langle \tilde{v}_{n-1}, B\tilde{v}_n\rangle \\ &\le 2k\langle \tilde{v}_{n+1}, B\tilde{v}_n\rangle + 2k\langle \tilde{v}_{n-1}, B\tilde{v}_n\rangle, \end{aligned}$$

i.e.

$$\begin{aligned} L_{n+1} &=: |\tilde{v}_{n+1}|^2 + |\tilde{v}_n|^2 - 2k \text{ Re } \langle \tilde{v}_{n+1}, B\tilde{v}_n \rangle \\ &\leq |\tilde{v}_n|^2 + |\tilde{v}_{n-1}|^2 - 2k \text{ Re } \langle \tilde{v}_n, B\tilde{v}_{n-1} \rangle = L_n. \end{aligned}$$

Therefore

$$L_n \leq L_1.$$

Observing that

$$\langle v, Bw \rangle \leq \tfrac{1}{2}|B|(|v|^2 + |w|^2)$$

(3.16) follows. □

The last theorem tells us that the combination of leap-frog and the midpoint rule is stable, if the slow part is oscillatory ($B = -B^*$). If this is not the case, then the weak instability of the leap-frog scheme can cause difficulties.

It is very desirable to prove stability of other combinations, for example, the combination of the implicit Euler scheme with a Runge–Kutta method . Also, it is important to investigate when such a combination automatically determines the slow solution.

Richardson extrapolation. As in the previous section the method depends only on the existence of an asymptotic expansion. Therefore it can be used here.

3.4. Highly oscillatory solutions of linear systems

Highly oscillatory problems have been studied for a long time, and a large number of perturbation techniques have been developed: multi-scaling, averaging and the near identity transformation (see, for example, Bogoliubov and Mitropolsky (1961), Nayfeh (1973), Hoppensteadt and Miranker (1976), Kevorkian and Cole (1981), Neu (1980)). For the most part these tools are difficult to implement numerically. We feel that effective numerical techniques are only available for special problems. We will discuss such methods in the next two sections.

In this section we consider linear systems

$$\begin{aligned} \varepsilon y' &= A(t)y, \quad 0 < \varepsilon \leq \varepsilon_0, \\ y(0) &= y_0. \end{aligned} \tag{3.17}$$

We make

Assumption 3.1 $A(t), A^{-1}(t) \in C^\infty$ and their derivatives are bounded independently of ε. The eigenvalues of A are distinct and

$$\text{Re } \lambda \leq 0.$$

If one solves (3.17) by difference approximation, then one has to use a time step $k \ll \varepsilon$. We want to show that one can solve the system by analytic means.

This assumption implies that there is a nonsingular transformation

$$S(t) = (s_1(t), \dots, s_n(t)) \in C^\infty, \quad s_j(t) \text{ eigenvalues of } A,$$

varying on the slow scale such that

$$S^{-1}AS = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} =: \Lambda.$$

Introducing into (3.17) a new variable $y_1 = S(t)y$ gives us

$$\varepsilon \, \mathrm{d}y_1/\mathrm{d}t = (\Lambda + \varepsilon B)y_1, \quad B = -S^{-1} \, \mathrm{d}S/\mathrm{d}t.$$

Now we can find a slowly varying transformation S_1 such that

$$(I+\varepsilon S_1)^{-1}(\Lambda+\varepsilon B)(I+\varepsilon S_1) = \begin{pmatrix} \lambda_1 + \varepsilon\lambda_{11} & & 0 \\ & \ddots & \\ 0 & & \lambda_n + \varepsilon\lambda_{n1} \end{pmatrix} =: \Lambda+\varepsilon\Lambda_1.$$

The change of variables

$$y_1 = (I + \varepsilon S_1)y_2$$

gives us

$$\mathrm{d}y_2/\mathrm{d}t = (\Lambda + \varepsilon\Lambda_1 + \varepsilon^2 B_1)y_2.$$

This process can be continued. Thus we can diagonalize the system to any order of $\mathcal{O}(\varepsilon^p)$. Neglecting the $\mathcal{O}(\varepsilon^p)$-terms we obtain scalar equations, whose solutions can be written down explicitly. We have proved

Theorem 3.1 The solution of (3.17) can be calculated analytically to any order in ε.

If the eigenvalues change multiplicity, then difficulties arise. An initial discussion can be found in Scheid (1982).

We consider now systems

$$\mathrm{d}y/\mathrm{d}t = A(t)y + F(t, \varepsilon),$$

where A is slowly varying and F has the property that

$$\int_0^t F(\eta, \varepsilon) \, \mathrm{d}\eta = \mathcal{O}(\varepsilon).$$

For example, this is the case if

$$F = \mathrm{e}^{\mathrm{i}t/\varepsilon} g(t), \quad g(t) \text{ slowly varying.}$$

By Duhamel's principle we can write the solution as

$$y(t) = S(t,0)y(0) + \int_0^t S(t,\xi)F(\xi)\,\mathrm{d}\xi,$$

where $S(t,\xi)$ denotes the solution operator of the homogeneous differential equation. It is a slowly vaying function of t,ξ. Therefore integration by parts gives us

$$\int_0^t S(t,\xi)F(\xi)\,\mathrm{d}\xi = S(t,\xi)\int_0^\xi F(\eta)\,\mathrm{d}\eta\Big|_0^t - \int_0^t \frac{\partial S(t,\xi)}{\partial \xi}\int_0^\xi F(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi = \mathcal{O}(\varepsilon).$$

Thus F changes the solution by an $\mathcal{O}(\varepsilon)$-term. One can derive an asymptotic expansion, if more about F is known.

Numerical methods based on these results are exploited in Amdursky and Ziv (1977), Fatunla (1980), Gautschi (1961), Miranker (1981) and Scheid (1982).

3.5. Highly oscillatory solutions of nonlinear equations

We start with a number of examples.

$$\begin{aligned} y' &= \frac{\mathrm{i}\lambda}{\varepsilon}y + y^2, \\ y(0) &= y_0. \end{aligned} \tag{3.18}$$

We can calculate the solution of (3.11) explicitly. Introducing a new variable by

$$y = \exp\left(\frac{\mathrm{i}\lambda}{\varepsilon}t\right)\tilde{y}$$

gives us

$$\begin{aligned} \tilde{y}' &= \exp\left(\frac{\mathrm{i}\lambda}{\varepsilon}t\right)\tilde{y}^2, \\ \tilde{y}(0) &= y_0. \end{aligned} \tag{3.19}$$

Therefore

$$\int_0^t \frac{\tilde{y}'}{\tilde{y}^2}\mathrm{d}t = -\frac{\mathrm{i}\varepsilon}{\lambda}\left[\exp\left(\frac{\mathrm{i}\lambda}{\varepsilon}t\right) - 1\right],$$

i.e.

$$\begin{aligned} \tilde{y} &= \frac{1}{\frac{1}{y_0} + \frac{\mathrm{i}\varepsilon}{\lambda}\left[\exp\left(\frac{\mathrm{i}\lambda}{\varepsilon}t\right) - 1\right]} = \frac{y_0}{1 + \frac{\mathrm{i}\varepsilon}{\lambda}y_0\left[\exp\left(\frac{\mathrm{i}\lambda}{\varepsilon}t\right) - 1\right]} \\ &= y_0\left\{1 - \frac{\mathrm{i}\varepsilon}{\lambda}y_0\left[\exp\left(\frac{\mathrm{i}\lambda}{\varepsilon}t\right) - 1\right]\right\} + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{3.20}$$

Thus the nonlinear term changes the solution only by $\mathcal{O}(\varepsilon)$ in arbitrarily long time intervals.

It is also useful to calculate the solution in another way. (3.17) gives us

$$\begin{aligned}\tilde{y} - y_0 &= \int_0^t \exp\left(\frac{i\lambda}{\varepsilon}\xi\right)\tilde{y}^2\,d\xi \\ &= -\frac{i\varepsilon}{\lambda}\exp\left(\frac{i\lambda}{\varepsilon}\xi\right)\tilde{y}^2\Big|_0^t + \frac{2i\varepsilon}{\lambda}\int_0^t \exp\left(\frac{i\lambda}{\varepsilon}\xi\right)\tilde{y}\tilde{y}'\,d\xi \\ &= \frac{i\varepsilon}{\lambda}\left[\exp\left(\frac{i\lambda}{\varepsilon}t\right)\tilde{y}^2(t) - \tilde{y}^2(0)\right] + \frac{2i\varepsilon}{\lambda}\int_0^t \exp\left(\frac{2i\lambda}{\varepsilon}\xi\right)\tilde{y}^3\,d\xi.\end{aligned}$$

The last integral can again be treated by integration by parts. Therefore

$$\tilde{y}(t) + \frac{i\varepsilon}{\lambda}\exp\left(\frac{i\lambda}{\varepsilon}t\right)\tilde{y}^2(t) = y_0 + \frac{i\varepsilon}{\lambda}y^2(0) + \mathcal{O}(\varepsilon^2),$$

i.e.

$$\tilde{y} = y_0\left\{1 - \frac{i\varepsilon}{\lambda}y_0\left[\exp\left(\frac{i\lambda}{\varepsilon}t\right) - 1\right]\right\} + \mathcal{O}(\varepsilon^2),$$

and we again obtain (3.18).

Now consider

$$y' = \frac{i\lambda_1}{\varepsilon}y + v, \quad v' = \frac{i\lambda_2}{\varepsilon}v + v^2. \tag{3.21}$$

The change of variables

$$y = \exp\left(\frac{i\lambda_1}{\varepsilon}t\right)\tilde{y}, \quad v = \exp\left(\frac{i\lambda_2}{\varepsilon}t\right)\tilde{v}$$

gives us

$$\tilde{y}' = \exp\left(\frac{i(\lambda_2 - \lambda_1)t}{\varepsilon}\right)\tilde{v}, \quad \tilde{v}' = \exp\left(\frac{i\lambda_2}{\varepsilon}t\right)\tilde{v}^2.$$

By (3.19)

$$\tilde{v} = v_0 + \sum_{j=1}^{\infty}\varepsilon^j\beta^j\exp\left(\frac{ij\lambda_2}{\varepsilon}t\right).$$

Therefore

$$\tilde{y}' = \exp\left(\frac{i}{\varepsilon}(\lambda_2 - \lambda_1)t\right)v_0 + \sum_{j=1}^{\infty}\varepsilon^j\beta^j\exp\left(\frac{i}{\varepsilon}[(j+1)\lambda_2 - \lambda_1]t\right).$$

If $\nu\lambda_2 - \lambda_1 \neq 0$ for all $\nu = 1, 2, \ldots$, then

$$\tilde{y}(t) = y_0 + \mathcal{O}(\varepsilon).$$

However, if $\nu\lambda_2 = \lambda_1$, then resonance occurs and

$$\tilde{y}(t) = \begin{cases} y(0) + \varepsilon^{\nu-1}\beta^{\nu-1}t & \text{if } \nu > 1, \\ y(0) + tv_0 & \text{if } \nu = 1. \end{cases}$$

Thus the solution is not bounded.

Now we can discuss systems

$$\begin{aligned} y' &= \frac{\mathrm{i}}{\varepsilon}\Lambda y + P(y), \\ y(0) &= y_0, \end{aligned} \tag{3.22}$$

where P is a polynomial in y. Introducing new variables by

$$y = \exp\left(\frac{\mathrm{i}}{\varepsilon}\Lambda t\right)\tilde{y}$$

gives us

$$\begin{aligned} \tilde{y}' &= \exp\left(-\frac{\mathrm{i}}{\varepsilon}\Lambda t\right) P\left[\exp\left(\frac{\mathrm{i}}{\varepsilon}\Lambda t\right)\tilde{y}\right], \\ \tilde{y}(0) &= y_0. \end{aligned} \tag{3.23}$$

The right-hand side of (3.22) consists of expressions

$$\exp\left[\frac{\mathrm{i}}{\varepsilon}\left(\sum \alpha_j\lambda_j\right)t\right] p(\tilde{y}), \tag{3.24}$$

where the α_j are integers and p is a polynomial in y. There are two possibilities.

1 $\tau = \sum \alpha_j\lambda_j = 0$ **for some terms.** In this case (3.22) has the form

$$\tilde{y}' = Q_0(\tilde{y}) + Q_1(\tilde{y}), \tag{3.25}$$

where Q_0, Q_1 contain the terms with and without exponentials, respectively. Our result in the last section tells us that we commit an error of order $\mathcal{O}(\varepsilon)$, if we neglect Q_1. Thus y is to first approximation the solution of

$$\tilde{y}' = Q_1(\tilde{y}), \quad \tilde{y}(0) = y_0, \tag{3.26}$$

i.e. in general $\tilde{y}(t)$ does not stay close to y_0.

2 $\tau = \sum \alpha_j\lambda_j \neq 0$ **for all terms.** Integration by parts gives us

$$\begin{aligned} \tilde{y}(t) &= y_0 + \sum_\tau \int_0^t \exp\left(\frac{\mathrm{i}\tau}{\varepsilon}\xi\right) p_\tau(\tilde{y})\,\mathrm{d}\xi \\ &= y_0 - \mathrm{i}\varepsilon \sum_\tau \frac{1}{\tau}\exp\left(\frac{\mathrm{i}\tau}{\varepsilon}\xi\right) p_\tau(\tilde{y})\Big|_0^t + \mathrm{i}\varepsilon \sum_\tau \frac{1}{\tau}\int_0^t \exp\left(\frac{\mathrm{i}\tau}{\varepsilon}\xi\right)\frac{\partial p_\tau}{\partial \tilde{y}}\tilde{y}'\,\mathrm{d}\xi \\ &= y_0 - \mathrm{i}\varepsilon \sum_\tau \frac{1}{\tau}\exp\left(\frac{\mathrm{i}\tau}{\varepsilon}\xi\right) p_\tau(\tilde{y})\Big|_0^t + \mathrm{i}\varepsilon \sum_\tau \frac{1}{\tau}\int_0^t \exp\left(\frac{\mathrm{i}\tau}{\varepsilon}\xi\right)\tilde{p}_\tau(\tilde{y})\,\mathrm{d}\xi. \end{aligned} \tag{3.27}$$

The integrals in (3.26) are over terms of type (3.23) and therefore we can repeat the previous arguments. If some of the terms are not of exponential type, then they will in general be of order $\mathcal{O}(\varepsilon t)$.

If all the terms are of exponential type, then we can use integration by parts to reduce them to (at least) $\mathcal{O}(\varepsilon^2 t)$.

We obtain

Theorem 3.2 Assume that for all integers α_j the linear combinations $\sum \alpha_j \lambda_j$ do not vanish. Then

$$\tilde{y} = y_0 + \mathcal{O}(\varepsilon)$$

in time intervals $0 \leq t \leq T$. $T = \mathcal{O}(\varepsilon^p)$ for any p.

There are no difficulties in extending the results and techniques to more general equations

$$y' = \frac{1}{\varepsilon}\Lambda(t)y + P(y,t).$$

Here $\Lambda(t)$ is slowly varying and $P(y,t)$ is a polynomial in y with slowly varying coefficients in time.

The numerical methods based on these results are exploited in Kreth (1977), Miranker and Wabba (1976), Miranker and van Veldhuisen (1978) and Scheid (1982).

3.6. Calculations of solutions, which contain both a fast and a slow part

One can solve these problems by brute force, i.e. use a time step $k \ll \varepsilon$. If one instead wants to calculate only the slow scale, i.e. $\varepsilon \ll k \ll 1$, then one has to combine analytic techniques with numerical methods. Very little is known about how to do this, see however (Petzold, 1981) for a different approach.

If the system has the form (2.1), splitting techniques have been used: Assume that we know the solution at time t_n, then we calculate the solution of

$$\begin{aligned} (w^{(1)}(t))' &= f_1(w^{(1)}, t), \quad t_n \leq t \leq t_{n+1}, \\ w^{(1)}(t_n) &= w(t_n) \end{aligned}$$

at $t = t_{n+1}$ to obtain $w^{(1)}(t_{n+1})$.

The next step is to solve

$$\begin{aligned} (w(t))' &= \frac{1}{\varepsilon}A(t)w(t), \quad t_n \leq t \leq t_{n+1}, \\ w(t_n) &= w^{(1)}(t_{n+1}) \end{aligned}$$

analytically, using the results of Section 3.3. This gives us $w(t_{n+1})$. It is not at all clear what the accuracy of this procedure is. We believe it has to be modified before it is generally useful, because in general the error is $\mathcal{O}(1)$.

Assume now that the system has the form (2.2). If the fast part of the

solution is small, then by Theorem 2.2 the effect of the fast part on the slow scale is one order of magnitude smaller. Therefore we can calculate the slow part of the solution first, and then treat the fast part as a perturbation, i.e. we have to solve

$$\varepsilon(y^{\mathrm{F}})' = (A(t) + \varepsilon C_1(v^{\mathrm{S}}, y^{\mathrm{S}}, t))y^{\mathrm{F}}$$

by analytic techniques as described in Section 3.3. The next step is to determine the effect of the fast scale on the slow scale. We believe that progress can be made along these lines, but no results are available yet.

4. Partial differential equations

4.1. General theory

Let $0 < \varepsilon \leq \varepsilon_0$ be a small constant, $x = (x_1, \ldots, x_s)$ be a point in the real s-dimensional Euclidian space R_s, e_j the unit vector in the x_j direction and $u = (u^{(1)}(x,t), \ldots, u^{(n)}(x,t))^T$ a vector function with n components. We consider systems

$$u_t = \varepsilon^{-1} P_0(\partial/\partial x)u + P_1(u, x, t, \varepsilon, \partial/\partial x)u + F(x,t) \tag{4.1}$$

with periodic boundary conditions

$$u(x + 2\pi e_j) = u(x), \quad j = 1, 2, \ldots, s$$

and smooth periodic initial data

$$u(x, 0) = f(x). \tag{4.2}$$

Here $F(x,t)$ is a smooth function of x, t with derivatives of order $\mathcal{O}(1)$, and the coefficients of

$$\begin{aligned} P_0 &= \sum_{j=1}^{s} A_j \partial/\partial x_j, \quad A_j = A_j^* \text{ constant matrices,} \\ P_1 &= \sum_{j=1}^{s} B_j(u, x, t, \varepsilon)\partial/\partial x_j, \quad B_j = B_j^* \text{ smooth functions of all variables,} \end{aligned} \tag{4.3}$$

are real symmetric matrices.

We want to prove that the bounded derivative principle is valid. We follow Browning and Kreiss (1982) closely. (See also Klainerman and Majda (1982) and Kreiss (1980).)

Theorem 4.1 Assume that p time derivatives at $t = 0$ are bounded independently of ε. Then the same is true in a time interval $0 \leq t \leq T$, where $T > 0$ does not depend on ε.

Proof. We consider first the system

$$w_t = P_1(w, x, t, \varepsilon, \partial/\partial x)w. \tag{4.4}$$

Let

$$(u, v) = \int \langle u, v \rangle \mathrm{d}x, \quad \|u\|^2 = (u, u)$$

denote the usual L_2-scalar product and norm. In the usual way we can derive *a priori* estimates by constructing differential inequalities for

$$\frac{\partial}{\partial t}\|D^{|j|}w\|^2 = 2\,\mathrm{Re}\,(D^{|j|}w, D^{|j|}(P_1 w)), \tag{4.5}$$

where $j = (j_1, \ldots, j_s)$, $|j| = \sum j_i$, is a multi-index, and

$$D^{|j|}w = \partial^{j_1}/\partial x_1^{j_1}, \ldots, \partial^{j_s}/\partial x_s^{j_s}.$$

(See, for example, Kreiss and Lorenz (1989).)

If we consider all expressions $\|D^{|j|}w\|^2$ with $|j| \leq [s/2] + 2$, then we can obtain a closed system of differential inequalities. The solutions of this system are bounded in some time interval $0 \leq t \leq T$, where $T > 0$ depends on the initial data but not on ε. Thus we obtain bounds of the first $[s/2]+2$ space derivatives. Higher order derivatives can then be estimated in the same time interval.

Now consider the system (4.1). Corresponding to (4.5) we have

$$\frac{\partial}{\partial t}\|D^{|j|}u\|^2 = 2\varepsilon^{-1}\,\mathrm{Re}\,(D^{|j|}u, P_0(\partial/\partial x)D^{|j|}u) + 2\,\mathrm{Re}\,(D^{|j|}u, D^{|j|}(P_1 u)). \tag{4.6}$$

P_0 is a first-order operator with constant symmetric matrix coefficients. Thus

$$2\,\mathrm{Re}\,(D^{|j|}u, P_0(\partial/\partial x)D^{|j|}u) \equiv 0,$$

and we also obtain for u the relationships (4.5). Therefore we can estimate all derivatives independently of ε in the same time interval, where we can estimate the derivatives $D^{|j|}w$. In particular, if we can estimate $D^{|j|}w$ for all times, then the same is true for u.

To obtain estimates of time derivatives we differentiate (4.1) with respect to t. $v = u_t$ satisfies

$$v_t = \varepsilon^{-1}P_0(\partial/\partial x)v + P_1(u, x, t, \varepsilon, \partial/\partial x)v + F_1.$$

Here F_1 depends on x, t and on u and its derivatives. Therefore $\mathrm{Re}\,(v, P_0 v) = 0$ implies

$$\begin{aligned}\frac{\partial}{\partial t}\|v\|^2 &= 2\varepsilon^{-1}\,\mathrm{Re}(v, P_0(\partial/\partial x)v) + (v, P_1 v) + (v, F_1) \\ &\leq \text{constant} \times (\|v\|^2 + \|F_1\|^2).\end{aligned}$$

Thus, if $v(x,0)$ is bounded independently of ε, then v stays bounded as long as the space derivatives stay bounded. Higher time derivatives are estimated correspondingly. This proves the theorem. □

We will now make the connection with the theory for ordinary differential equations. For simplicity we assume that the coefficients of P_1 are polynomials in u and do not depend on x, t explicitly.

We have seen that for smooth initial data the solution of the differential equation is smooth in space. Therefore we can develop it into a rapidly convergent Fourier series

$$u(x,t) = \sum_{\omega} \hat{u}(\omega,t) \exp(\mathrm{i}\langle\omega,x\rangle), \quad \langle\omega,x\rangle = \sum_{j} \omega_j x_j. \tag{4.7}$$

Introducing (4.7) into (4.1) and neglecting all frequencies with $|\omega| > N$ gives a system of ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}(\omega,t) = \varepsilon^{-1} P_0(\mathrm{i}\omega)\hat{u}(\omega,t) + \hat{G}_\omega(\hat{u},t). \tag{4.8}$$

$P_0(\mathrm{i}\omega)$ is a skew Hermitean matrix and therefore there is a unitary matrix $U(\mathrm{i}\omega)$ such that

$$U^*(\mathrm{i}\omega)P_0(\mathrm{i}\omega)U(\mathrm{i}\omega) = \begin{pmatrix} \hat{R}(\mathrm{i}\omega) & 0 \\ 0 & 0 \end{pmatrix}, \quad \det|\hat{R}| \neq 0. \tag{4.9}$$

Thus we can transform (4.8) into the form (2.2).

We will now formalize the process. We make

Assumption 4.1 There is a constant $\delta > 0$ such that for all ω

$$|\hat{R}^{-1}(\mathrm{i}\omega)| \leq \delta^{-1}.$$

L_2 consists of all functions

$$f = \sum_{\omega} \hat{f}(\omega) \exp(\mathrm{i}\langle\omega,x\rangle), \quad \sum |\hat{f}(\omega)|^2 < \infty,$$

which can be expanded into a Fourier series. Let Q denote the projection

$$f^I = Qf = \sum_{\omega} \hat{U}(\mathrm{i}\omega) \begin{pmatrix} I_\omega & 0 \\ 0 & 0 \end{pmatrix} \hat{U}^*(\mathrm{i}\omega) \exp(\mathrm{i}\langle\omega,x\rangle)\hat{f}(\omega).$$

Here I_ω is the unit matrix of the same dimension as $\hat{R}(\mathrm{i}\omega)$, Q splits L_2 into two subspaces $L_2^{\mathrm{I}}, L_2^{\mathrm{II}}$ defined by

$$f^{\mathrm{I}} = Qf, \quad f^{\mathrm{II}} = (I-Q)f, \quad f = f^{\mathrm{I}} + f^{\mathrm{II}}.$$

Note that Q commutes with P_0, i.e. $QP_0 = P_0Q$ because

$$QP_0u = \sum_{\omega} \hat{U}(\mathrm{i}\omega) \begin{pmatrix} I_\omega & 0 \\ 0 & 0 \end{pmatrix} \hat{U}^*(\mathrm{i}\omega)\hat{U}(\mathrm{i}\omega) \begin{pmatrix} \hat{R}(\mathrm{i}\omega) & 0 \\ 0 & 0 \end{pmatrix}$$

$$
\begin{aligned}
&\times U^*(\mathrm{i}\omega)\hat{u}(\omega)\exp(\mathrm{i}\langle\omega,x\rangle)\\
&= \sum_\omega \hat{U}(\mathrm{i}\omega)\begin{pmatrix}\hat{R}(\mathrm{i}\omega) & 0\\ 0 & 0\end{pmatrix}\hat{U}^*(\mathrm{i}\omega)\hat{u}(\omega)\exp(\mathrm{i}\langle\omega,x\rangle)\\
&= \sum_\omega \hat{U}(\mathrm{i}\omega)\begin{pmatrix}\hat{R}(\mathrm{i}\omega) & 0\\ 0 & 0\end{pmatrix}\hat{U}^*(\mathrm{i}\omega)\hat{U}(\mathrm{i}\omega)\begin{pmatrix}I_\omega & 0\\ 0 & 0\end{pmatrix}\\
&\quad\times \hat{U}^*(\mathrm{i}\omega)\hat{u}(\omega)\exp(\mathrm{i}\langle\omega,x\rangle)\\
&= P_0Qu.
\end{aligned}
$$

Also,

$$
\begin{aligned}
P_0u^{\mathrm{II}} &= P_0(I-Q)u\\
&= \sum_\omega \hat{U}(\mathrm{i}\omega)\begin{pmatrix}\hat{R}(\mathrm{i}\omega) & 0\\ 0 & 0\end{pmatrix}\left[I-\begin{pmatrix}I_\omega & 0\\ 0 & 0\end{pmatrix}\right]\hat{U}^*(\mathrm{i}\omega)\hat{u}(\omega)\exp(\mathrm{i}\langle\omega,x\rangle)\\
&= 0.
\end{aligned}
$$

We can define the inverse of P_0 on L_2^{I} uniquely by

$$
P_0^{-1}u^{\mathrm{I}} = \sum_\omega \hat{U}(\mathrm{i}\omega)\begin{pmatrix}\hat{R}^{-1}(\mathrm{i}\omega) & 0\\ 0 & 0\end{pmatrix}\hat{U}^*(\mathrm{i}\omega)\hat{u}(\omega)\exp(\mathrm{i}\langle\omega,x\rangle),
$$

and (4.9) gives us

$$
\|P_0^{-1}u^{\mathrm{I}}\| \le \delta^{-1}\|u^{\mathrm{I}}\|. \tag{4.10}
$$

Using the projection Q, we can now write the system (4.1) in the form

$$
u_t^{\mathrm{I}} = \varepsilon^{-1}P_0u^{\mathrm{I}} + (P_1(u,\partial/\partial x)u)^{\mathrm{I}} + F^{\mathrm{I}}, \tag{4.11a}
$$

$$
u_t^{\mathrm{II}} = (P_1(u,\partial/\partial x)u)^{\mathrm{II}} + F^{\mathrm{II}}, \quad u = u^{\mathrm{I}} + u^{\mathrm{II}}, \tag{4.11b}
$$

which is the generalization of (2.2) to a partial differential equation.

We can now show that if u is slow to order p, then u^{I} is determined by u^{II} up to terms of order $\mathcal{O}(\varepsilon^p)$. $\partial u/\partial t$ is bounded independently of ε if and only if

$$
P_0u^{\mathrm{I}} = \mathcal{O}(\varepsilon),
$$

hence

$$
u^{\mathrm{I}} = \varepsilon u_1^{\mathrm{I}}, \quad u_1^{\mathrm{I}} = \mathcal{O}(1), \tag{4.12}
$$

i.e. u has to first approximation no component in L_2^{I}. Therefore, to first approximation, the solution to our problem is given by

$$
u^{\mathrm{I}} \equiv 0, \quad u_t^{\mathrm{II}} = (P_1(u^{\mathrm{II}},\partial/\partial x)u^{\mathrm{II}})^{\mathrm{II}} + F^{\mathrm{II}}.
$$

Differentiating (4.11) with respect to t and assuming that (4.12) holds gives us

$$
u_{tt}^{\mathrm{I}} = \varepsilon^{-1}P_0u_t^{\mathrm{I}} + \mathcal{O}(1)
$$

$$= \varepsilon^{-1}P_0\Big(P_0u_1^{\mathrm{I}} + (P_1(u^{\mathrm{II}}, \partial/\partial x)u^{\mathrm{II}})^{\mathrm{I}} + F^I\Big) + \mathcal{O}(1),$$

$$u_{tt}^{\mathrm{II}} = \mathcal{O}(1). \tag{4.13}$$

Therefore the second time derivative is bounded independently of ε if and only if

$$P_0u_1^{\mathrm{I}} + (P_1(u^{\mathrm{II}}, \partial/\partial x)u^{\mathrm{II}})^{\mathrm{I}} + F^{\mathrm{I}} = \mathcal{O}(\varepsilon).$$

Thus u^{I} is determined by u^{II} up to terms of order $\mathcal{O}(\varepsilon^2)$. This process can be continued, and we obtain the desired relation between u^{I} and u^{II}.

As in Section 2 we can also derive the asymptotic expansion by the iteration

$$\begin{aligned}(u^{(n-1)})_t^{\mathrm{I}} &= \varepsilon^{-1}P_0(u^{(n)})^{\mathrm{I}} + (P_1(u^{(n)}, \partial/\partial x)u^{(n)})^{\mathrm{I}} + F^{\mathrm{I}}, \quad (u^{(-1)})^{\mathrm{I}} \equiv 0,\\ (u^{(n)})_t^{\mathrm{II}} &= (P_1(u^{(n)}, \partial/\partial x)u^{(n)})^{\mathrm{II}} + F^{\mathrm{II}}, \quad n = 0, 1, 2, \dots .\end{aligned} \tag{4.14}$$

It again gives us the relationship between u^{I} and u^{II}. For meteorological applications there are many papers, which describe how to obtain this relationship in practice. (See, for example, Kasahara (1982), Leith (1980), Machenhauer (1977).)

One can generalize the results considerably. However, the theory becomes much more complicated, if $P_0 = P_0(x, t, \partial/\partial x)$ depends on x, t or if one wants to treat the initial boundary value problem. Details can be found in Browning and Kreiss (1982), Kreiss (1980) and Tadmor (1982). Numerical methods are discussed in Guerra and Gustafsson (1982), Gustafsson (1980a), Gustafsson (1980b), Gustaffson and Kreiss (1983).

4.2. The wave equation

We consider in this section the wave equation written as a first-order system

$$\begin{aligned}\varepsilon u_t &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}u_x + F\\ u(x, 0) &= f.\end{aligned} \tag{4.15}$$

Here $u, F, f \in C^\infty$ are vector-valued 2π-periodic functions. (4.15) is of the form (4.3) with

$$P_0(\mathrm{i}\omega) = \mathrm{i}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\omega, \quad P_1 \equiv 0.$$

Thus the theory applies.

(4.15) implies

$$\varepsilon\frac{\partial}{\partial t}\int_0^{2\pi} u\,\mathrm{d}x = \int_0^{2\pi} F\,\mathrm{d}x,$$

i.e.

$$\int_0^{2\pi} u(x,t)\,dx - \int_0^{2\pi} u(x,0)\,dx = \frac{1}{\varepsilon}\int_0^t \int_0^{2\pi} F(x,\xi)\,dx\,d\xi.$$

If

$$\int_0^{2\pi} F\,dx = \mathcal{O}(1),$$

then the mean value of the solution becomes unbounded for $\varepsilon \to 0$. For simplicity we assume that

$$\int_0^{2\pi} f(x)\,dx = 0, \quad \int_0^{2\pi} F(x,t)\,dx \equiv 0. \tag{4.16}$$

Then

$$\int_0^{2\pi} u(x,t)\,dx \equiv 0. \tag{4.17}$$

We shall now derive the asymptotic expansion. We proceed in the same manner as for the ordinary diferential equations. u has one derivative bounded independently of ε, if

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u_x + F = \mathcal{O}(\varepsilon).$$

This suggests the substitution

$$u = u_1 + \varphi_0, \tag{4.18}$$

where φ_0 is the solution of

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \varphi_{0x} = -F, \quad \int_0^{2\pi} \varphi_0\,dx = 0.$$

Introducing (4.18) into (4.15) gives us

$$\begin{aligned} \varepsilon u_{1t} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u_{1x} - \varepsilon\varphi_{0t}, \quad \int_0^{2\pi} u_1\,dx = 0, \\ u_1(x,0) &= f(x) - \varphi_0(x,0). \end{aligned} \tag{4.19}$$

(4.19) is of the same form as (4.15) with the forcing reduced to $\mathcal{O}(\varepsilon)$. Therefore we can repeat the process and we obtain the slow solution

$$u^{\mathrm{S}} = \sum_{j=0}^{p-1} \varepsilon^j \varphi_j + \mathcal{O}(\varepsilon^p).$$

The fast part u^{F} is the solution of

$$\begin{aligned} \varepsilon v_t^{\mathrm{F}} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v_x^{\mathrm{F}}, \quad \int_0^{2\pi} v^{\mathrm{F}}\,dx = 0, \\ v^{\mathrm{F}}(x,0) &= f(x) - u^{\mathrm{S}}(x,0), \end{aligned} \tag{4.20}$$

i.e.

$$v^{\mathrm{F}}(x,t) = \sum_j a_j \begin{pmatrix} 1 \\ 1 \end{pmatrix} \exp\left[ij\left(x + \frac{t}{\varepsilon}\right)\right] + b_j \begin{pmatrix} 1 \\ -1 \end{pmatrix} \exp\left[ij\left(x - \frac{t}{\varepsilon}\right)\right],$$

where the a_j, b_j are determined by the initial data.

In applications the equations often do not appear as symmetric hyperbolic problems. As an example we consider instead of (4.15)

$$\begin{aligned} \begin{pmatrix} u \\ p \end{pmatrix}_t &= \begin{pmatrix} 0 & 1 \\ \varepsilon^{-2} & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix}_x + \begin{pmatrix} F \\ \varepsilon^{-2} G \end{pmatrix}, \\ u(x,0) &= f, \quad p(x,0) = g. \end{aligned} \tag{4.21}$$

We can symmetrize the equations by introducing a new variable

$$\tilde{p} = \varepsilon p$$

and obtain

$$\begin{aligned} \begin{pmatrix} u \\ \tilde{p} \end{pmatrix}_t &= \frac{1}{\varepsilon} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ \tilde{p} \end{pmatrix}_x + \begin{pmatrix} F \\ \varepsilon^{-1} G \end{pmatrix}, \\ u(x,0) &= f, \quad \tilde{p}(x,0) = \varepsilon g, \end{aligned} \tag{4.22}$$

which is of the same form as (4.15). Therefore we can again write down the asymptotic expansion of the slow part of the solution and obtain

$$u^{\mathrm{S}} = \varphi_0^{(1)} + \varepsilon \varphi_1^{(1)} + \cdots, \quad \tilde{p}^{\mathrm{S}} = \varepsilon \varphi_0^{(2)} + \varepsilon^2 \varphi_0^{(2)} + \cdots,$$

i.e. we also have bounded asymptotic expansions in the original variables $u, p = \varepsilon^{-1}\tilde{p}$. The fast part of the solution is again determined by the homogeneous equation (4.20) with initial data

$$u^{\mathrm{F}}(x,0) = f(x) - u^{\mathrm{S}}(x,0), \quad \tilde{p}^{\mathrm{F}}(x,0) = \varepsilon g(x) - \tilde{p}^{\mathrm{S}}(x,0).$$

Now we assume that the data have not been initialized. Then

$$u^{\mathrm{F}}(x,0) = \mathcal{O}(1), \quad \tilde{p}^{\mathrm{F}}(x,0) = \mathcal{O}(\varepsilon).$$

However, at later times energy from u^{F} will be transferred to $\tilde{p}^{\mathrm{F}}$ and therefore

$$u^{\mathrm{F}}(x,t) = \mathcal{O}(1), \quad \tilde{p}^{\mathrm{F}}(x,t) = \mathcal{O}(1).$$

Then we obtain in the original variables

$$u^{\mathrm{F}}(x,t) = \mathcal{O}(1), \quad p(x,t) = \varepsilon^{-1} p^{\mathrm{F}}(x,t) = \mathcal{O}(\varepsilon^{-1}),$$

and the amplitude of p will be amplified by a factor ε^{-1}.

For moderate values of ε this is not a problem. However, if ε becomes very small, it can cause a lot of trouble in numerical calculations. For example:

1 If the data are initialized analytically but the problem is solved numerically, then the initialization of the difference approximation is, due to truncation errors, different from the analytic initialization.

2 Rapid time changes in F can trigger large waves on the fast scale.
3 If ε is very small, then rounding errors can also cause difficulties.

Numerical methods. If one is only interested in the slow solution, then using the asymptotic expansion or Richardson extrapolation are efficient methods. (Observe that one uses Richardson extrapolation with moderate values of ε and therefore one can also treat nonsymmetric systems.)

If one is interested in both the fast and the slow part of the solution, then one can use asymptotic expansion for the slow part of the solution and solve (4.20) analytically.

5. Applications

5.1. Low Mach number flow

A slightly simplified version of the Euler equations for low Mach number flow in two space dimensions is given by

$$\begin{aligned} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= F \\ M^2(p_t + (\mathbf{u}\cdot\nabla)p) + \nabla\cdot u &= G \end{aligned} \tag{5.1}$$

with initial data

$$\mathbf{u}(x,y,0) = \mathbf{f}(x,y), \quad p(x,y,0) = g(x,y). \tag{5.2}$$

Here $0 < M^2 \ll 1$ is the Mach number and

$$\mathbf{u} = (u(x,y,t), v(x,y,t)), \quad p = p(x,y,t)$$

denote the velocity field and the pressure, respectively. We are interested in 2π-periodic solutions.

We can also write (5.1) in component form

$$\begin{pmatrix} u \\ v \\ \tilde{p} \end{pmatrix}_t + \begin{pmatrix} u & 0 & M^{-1} \\ 0 & u & 0 \\ M^{-1} & 0 & u \end{pmatrix} \begin{pmatrix} u \\ v \\ \tilde{p} \end{pmatrix}_x + \begin{pmatrix} v & 0 & 0 \\ 0 & v & M^{-1} \\ 0 & M^{-1} & v \end{pmatrix} \begin{pmatrix} u \\ v \\ \tilde{p} \end{pmatrix}_y = \begin{pmatrix} F_1 \\ F_2 \\ g \end{pmatrix}. \tag{5.3}$$

We have introduced

$$\tilde{p} = Mp \tag{5.4}$$

as a new variable such that the system is symmetric hyperbolic. Thus (5.1) has the same difficulties as (4.21).

The symbol of the large part

$$P_0(\mathrm{i}\omega) = M^{-1} \begin{pmatrix} 0 & 0 & \mathrm{i}\omega_1 \\ 0 & 0 & \mathrm{i}\omega_2 \\ \mathrm{i}\omega_1 & \mathrm{i}\omega_2 & 0 \end{pmatrix} \tag{5.5}$$

has rank two. The general theory tells us that there is one slow variable. In

the general theory the slow variable can only be defined via Fourier transform. However, in this case we can also identify it in physical space. It is the vorticity $\nabla \times u =: v_x - u_y$, because differentiating the second equation of (5.3) with respect to x and the first with respect to y and subtracting them gives us

$$\xi_t + u\xi_x + v\xi_y + (u_x + v_y)\xi = F_{2x} - F_{1y}. \tag{5.6}$$

The fast variables are the pressure and the dilutation $\nabla \cdot \mathbf{u} =: u_x + v_y$.

We will now describe the results in Kreiss *et al.* (1991). (See also Klainerman and Majda (1982).) We derive an asymptotic expansion of the slow part, starting from (5.1). If the derivatives are bounded independently of M, then the leading term must satisfy

$$\begin{aligned} \mathbf{U}_t + (\mathbf{U} \cdot \nabla)\mathbf{U} + \nabla P &= F \\ \nabla \cdot \mathbf{U} &= G \end{aligned} \tag{5.7}$$

with initial data defined by

$$\nabla \cdot \mathbf{U}(x, y, 0) = G(x, y, 0), \quad \nabla \times \mathbf{U}(x, y, 0) = \nabla \times \mathbf{f}(x, y). \tag{5.8}$$

Defining new variables by

$$\mathbf{u} = \mathbf{U} + \mathbf{u}', \quad p = P + p', \tag{5.9}$$

we obtain from (5.1)

$$\begin{aligned} \mathbf{u}'_t + (\mathbf{U} \cdot \nabla)\mathbf{u}' + (\mathbf{u}' \cdot \nabla)\mathbf{U} + (\mathbf{u} \cdot \nabla)\mathbf{u}' + \nabla p' &= 0 \\ M^2\left(p'_t + (\mathbf{U} \cdot \nabla)p' + (\mathbf{u}' \cdot \nabla)P + (\mathbf{u}' \cdot \nabla)p'\right) + \nabla \cdot \mathbf{u}' &= M^2 G_1 \end{aligned} \tag{5.10}$$

with

$$G_1 = -(p_t + (\mathbf{U} \cdot \nabla)P).$$

First we determine the slow part of $\mathbf{u}', p'$ and write

$$\mathbf{u}' = M^2\mathbf{U}_1 + \mathbf{u}'', \quad p' = M^2 P_1 + p'', \tag{5.11}$$

where

$$\begin{aligned} \mathbf{U}_{1t} + (\mathbf{U} \cdot \nabla)\mathbf{U}_1 + (\mathbf{U}_1 \cdot \nabla)\mathbf{U} + \nabla P_1 &= 0 \\ \nabla \cdot \mathbf{U}_1 &= G_1. \end{aligned} \tag{5.12}$$

The initial data for (5.11) are given by

$$\nabla \cdot \mathbf{U}_1(x, y, 0) = G_1(x, y, 0), \quad \nabla \times \mathbf{U}_1(x, y, 0) = 0.$$

Now we introduce $\mathbf{u}'', p''$ as new variables into (5.10) and repeat the procedure. We obtain

Theorem 5.1 We can expand the slow part of the solution of (5.1) into a

series

$$\begin{aligned} u &= \mathbf{U} + M^2\mathbf{U}_1 + \cdots + M^{2l}\mathbf{U}_l + u_{\mathrm{R}} =: \mathbf{U}^{(l)} + \mathbf{u}_{\mathrm{R}} \\ p &= P + M^2 P_1 + \cdots + M^{2l} P_l + p_{\mathrm{R}} = P^{(l)} + p_{\mathrm{R}}, \end{aligned}$$

where $\mathbf{U}_j, P_j$ satisfy linearized incompressible equations and their derivatives are bounded independently of ε. The remainder $\mathbf{u}_{\mathrm{R}}, p_{\mathrm{R}}$ are the solution of

$$\begin{aligned} &\mathbf{u}_{\mathrm{R}t} + (\mathbf{U}^{(l)} \cdot \nabla)\mathbf{u}_{\mathrm{R}} + (\mathbf{u}_{\mathrm{R}} \cdot \nabla)\mathbf{U}^{(l)} + (\mathbf{u}_{\mathrm{R}} \cdot \nabla)\mathbf{u}_{\mathrm{R}} + \nabla p_{\mathrm{R}} = M^{4l} F_{l+1} \\ &M^2\Big(p_{\mathrm{R}t} + (\mathbf{U}^{(l)} \cdot \nabla)p_{\mathrm{R}} + (\mathbf{u}_{\mathrm{R}} \cdot \nabla)P^{(l)} + (\mathbf{u}_{\mathrm{R}} \cdot \nabla)p_{\mathrm{R}}\Big) + \nabla \cdot \mathbf{u}_{\mathrm{R}} = M^{2l+2} G_l. \end{aligned} \tag{5.13}$$

The initial data

$$\mathbf{u}_{\mathrm{R}}(x, y, 0) = \mathbf{f}(x, y) - \mathbf{U}^{(l)}(x, y, 0), \quad p_{\mathrm{R}}(x, y, 0) = G(x, y, 0) - P^{(l)}(x, y, 0)$$

satisfy

$$\nabla \times \mathbf{u}_{\mathrm{R}}(x, y, 0) = 0.$$

One can prove (see Kreiss *et al.* (1991))

Theorem 5.2 In any finite time interval $0 \le t \le T$

$$\nabla \times \mathbf{u}_{\mathrm{R}}(x, y, t) = \mathcal{O}(M).$$

Thus $\mathbf{u}_{\mathrm{R}}, p_{\mathrm{R}}$ represent the fast part of the solution.

To discuss their behaviour and to simplify the notation we introduce new variables by

$$\tau = t/M, \quad q = M p_{\mathrm{R}}, \quad \mathbf{v} = \mathbf{u}_{\mathrm{R}}, \quad \mathbf{U}^{(l)} = \mathbf{U}, \quad P^{(l)} = P.$$

We also neglect the forcing. Then (5.13) becomes

$$\begin{aligned} &\mathbf{v}_\tau + M((\mathbf{U} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{U}) + \nabla q = 0 \\ &q_\tau + M(\mathbf{U} \cdot \nabla)q + M^2(\mathbf{v} \cdot \nabla)P + \nabla \cdot \mathbf{v} = 0, \\ &\mathbf{v}(x, y, 0) = \mathbf{f}(x, y) - \mathbf{U}(x, y, 0), \quad q(x, y, 0) = M(G(x, y, 0) - P(x, y, 0)). \end{aligned} \tag{5.14}$$

If we neglect terms multiplied by M, (5.14) becomes

$$\begin{aligned} \mathbf{v}_\tau + \nabla q &= 0 \\ q_\tau + \nabla \cdot \mathbf{v} &= 0. \end{aligned} \tag{5.15}$$

(5.14) can be rewritten as the wave equations for q and the dilutation $\nabla \cdot \mathbf{v}$. Here we see again that the amplitude of the fast waves are amplified by the factor $1/M$ because $p_{\mathrm{R}} = (1/M)q$. If $|\mathbf{v}(x, y, 0)| \approx 1$, then q grows on

the fast time scale from $\mathcal{O}(M)$ to $\mathcal{O}(1)$, i.e. p_R becomes $\mathcal{O}(1/M)$. Large fast waves can only be avoided, if $\mathbf{v}(x,y,0) = \mathcal{O}(M)$. Since the vorticity is $\mathcal{O}(M)$, this can only be achieved, if the dilutation $\nabla \cdot \mathbf{v}(x,y,0) = \mathcal{O}(M)$.

To solve this problem numerically we can use the asymptotic expansion. If we only want to determine the slow solutions, then we can also use Richardson extrapolation (see Johansson (1991)). If we are also interested in the fast part of the solution, then we have to solve (5.14). Here one should use (5.15) locally, because we can determine its solution analytically.

We have only discussed the periodic problem. However, one can also treat the initial boundary value problem (see Kreiss *et al.* (1991)).

5.2. Atmospheric motions

In this section we consider three-dimensional atmospheric motions and discuss results presented in Browning and Kreiss (1986, 1987). The corresponding results for oceanographic flows can be found in Browning *et al.* (1990). In Cartesian coordinates x, y, z are directed eastward, northward and upward, respectively, and the Eulerian equations have the form (see Kasahara (1974))

$$\begin{aligned}
&\mathrm{d}s/\mathrm{d}t = 0,\\
&\mathrm{d}/\mathrm{d}t = \partial/\partial t + u\,\partial/\partial x + v\,\partial/\partial y + w\,\partial/\partial z,\\
&\mathrm{d}p/\mathrm{d}t + \gamma p(u_x + v_y + w_z) = 0, \quad \gamma = 1.4,\\
&\quad \rho\,\mathrm{d}u/\mathrm{d}t + p_x - f\rho v = 0,\\
&\quad \rho\,\mathrm{d}v/\mathrm{d}t + p_y + f\rho u = 0, \quad \rho = sp^{-1}\gamma,\\
&\quad \rho\,\mathrm{d}w/\mathrm{d}t + p_z + \rho g = 0.
\end{aligned} \tag{5.16}$$

Here s is the entropy, p the pressure, ρ the density, u, v, w the velocity components in the x, y, z directions, respectively, and $g \approx 10\ \mathrm{m\ s^{-2}}$ the gravity acceleration. We make the β-plan approximation, i.e. the Coriolis force f is given by

$$f = 2\Omega(\sin\theta_0 + \gamma/r\ \cos\theta_0), \quad 2\Omega = 10^{-4}\ \mathrm{s^{-1}}, \quad r = 10^7\ \mathrm{m}, \tag{5.17}$$

where r is the radius, Ω the angular speed for the earth, and θ_0 the latitude of the coordinate origin.

We have to introduce scaled variables before we can apply our theory. We change the variables in such a way that the variables and their first derivatives are of order $\mathcal{O}(1)$.

$$\begin{aligned}
&x = L_1 x', \quad y = L_2 y', \quad z = Dy', \quad t = Tt',\\
&u = Uu', \quad v = Vv', \quad w = Ww'.
\end{aligned} \tag{5.18}$$

Density and pressure can be written in the form

$$p = P_0(p_0(z) + S_1 p'), \quad \rho = R_0(\rho_0(z) + S_1\rho'), \quad 0 < S_1 \ll 1, \tag{5.19}$$

where

$$P_0 \partial p_0/\partial z + g R_0 \rho_0 = 0,$$
$$P_0 = 10^5 \text{ kg m}^{-1} \text{ s}^{-2}, \quad R_0 = 1 \text{ kg m}^{-3}.$$

Equations (5.19) express the fact that a number of digits of the pressure and density are independent of x and y and that p and ρ are to the first approximation in the hydrostatic balance. Equations (5.18) also imply that

$$\begin{aligned} s &= R_0 P_0^{-1/\gamma} \rho_0(z)(p_0(z))^{-1/\gamma}(1 + S_1\rho'/\rho)(1 + S_1 p'/p_0)^{-1/\gamma} \\ &= R_0 P_0^{-1/\gamma} s_0(z)(1 + S_1 s'), \\ s_0(z) &= \rho_0(z)(p_0(z))^{-1/\gamma}, \quad s' = \rho'/\rho_0 - (1/\gamma)p'/p_0 + \mathcal{O}(S_1). \end{aligned} \tag{5.20}$$

We assume that the scales in the x, y directions are the same, that $\partial u/\partial t$ and $\partial v/\partial t$ balance the horizontal convection terms, and that the Coriolis force has a strong influence. This leads to the following relations:

$$\begin{aligned} U &= V, \quad L_1 = L_2 = L, \\ UT/L &= 1, \quad 2\Omega T = S_1 P_0/(R_0 U^2). \end{aligned} \tag{5.21}$$

These relationships are not valid for special types of motions like jet streams, ultralong waves and small-scale problems. For the treatment of these cases we refer to Browning and Kreiss (1986). Introducing the scaled variables into (5.16) gives us

$$\begin{aligned} &\frac{ds'}{dt'} + S_1^{-1} S_2 \tilde{s}(z)(1 + S_1 s')w' = 0, \\ &\frac{dp'}{dt'} + S_1^{-1} p_0 \left[\gamma\left(1 + \frac{S_1 p'}{p_0}\right)(u_x + v_y + S_2 w_z) + S_2 \tilde{p}(z) w'\right] = 0, \\ &\frac{du'}{dt'} + S_3 \left[\rho_0^{-1}\left(1 + \frac{S_1\rho'}{\rho_0}\right)^{-1} p'_{x'} - f' v'\right] = 0, \\ &\frac{dv'}{dt'} + S_3 \left[\rho_0^{-1}\left(1 + \frac{S_1\rho'}{\rho_0}\right)^{-1} p'_{y'} + f' u'\right] = 0, \\ &\frac{dw'}{dt'} + S_1^{-1} S_4 \rho_0^{-1}\left(1 + \frac{S_1\rho'}{\rho_0}\right)^{-1} (p'_{z'} - \gamma^{-1}\tilde{p}(z)\rho' + S_5 \rho_0 s' + \mathcal{O}(S_1)) = 0, \end{aligned} \tag{5.22}$$

where

$$\tilde{p}(z) = (\ln p_0)_z, \quad \tilde{s}(z) = (\ln s_0)_z;$$

typically,

$$\tilde{p} \approx -1.3, \quad -3 \le \tilde{s} \le -1;$$

$$d/dt = \partial/\partial t' + u' \partial/\partial x' + v' \partial/\partial y' + S_2 w' \partial/\partial z';$$

and the parameters S_i are given by

$$\begin{aligned} S_2 &= D^{-1}TW, \quad S_3 = 2\Omega T, \\ S_4 &= TP_0(DR_0W)^{-1}, \quad S_5 = gDP_0^{-1}R_0. \end{aligned} \tag{5.23}$$

We now choose the parameters according to the so-called large-scale dynamics:

$$L = 10^6 \text{ m}, \ D = 10^4 \text{ m}, \ U = V = 10 \text{ m s}^{-1}, \ S_1 = 10^{-2}, \ W = 10^{-2} \text{ m s}^{-1}. \tag{5.24}$$

Introducing these values into (5.23) and (5.24), we obtain, dropping the prime notation,

$$\begin{aligned} \frac{\mathrm{d}s}{\mathrm{d}t} + \tilde{s}(z)w &= 0, \\ \frac{\varepsilon^2}{\gamma p_0}\frac{\mathrm{d}p}{\mathrm{d}t} + u_x + v_y + \varepsilon Lw &= 0, \quad \varepsilon = 10^{-1}, \\ \varepsilon\frac{\mathrm{d}u}{\mathrm{d}t} + \rho_0^{-1}p_x - fv &= 0, \\ \varepsilon\frac{\mathrm{d}v}{\mathrm{d}t} + \rho_0^{-1}p_y + fu &= 0, \\ \rho_0\varepsilon^6\frac{\mathrm{d}w}{\mathrm{d}t} - L^*p + \rho_0 s &= 0. \end{aligned}$$

Here

$$\mathrm{d}/\mathrm{d}t = \partial/\partial t + u\,\partial/\partial x + v\,\partial/\partial y + \varepsilon w\,\partial/\partial z,$$
$$Lw = w_z + \gamma^{-1}\tilde{p}(z)w, \quad L^*p = -p_z + \gamma^{-1}\tilde{p}(z)p.$$

For simplicity only, we have neglected terms of order $\mathcal{O}(S_1)$. Also, by (5.17),

$$f = f_0 + \varepsilon\beta y.$$

We will only consider the mid-latitude case $f_0 \approx 1$.

For ease of discussion we simplify the equations slightly by replacing

$$\tilde{s}(z) \to -1, \quad (\gamma p_0)^{-1} \to 1, \quad f \to 1, \quad lw \to w_z, \quad \rho_0 \to 1, \quad L^*p \to p_z, \tag{5.25}$$

and obtain

$$\begin{aligned} \frac{\mathrm{d}s}{\mathrm{d}t} - w &= 0, \\ \varepsilon^2\frac{\mathrm{d}p}{\mathrm{d}t} + u_x + v_y + \varepsilon w_z &= 0, \\ \varepsilon\frac{\mathrm{d}u}{\mathrm{d}t} + p_x - v &= 0, \\ \varepsilon\frac{\mathrm{d}v}{\mathrm{d}t} + p_y + u &= 0, \end{aligned}$$

$$\eta \frac{dw}{dt} + p_z + s = 0. \tag{5.26}$$

Here we have replaced ε^6 by η, and we think of η as another small parameter.

We will now discuss the initialization. The first time derivatives are bounded independently of ε, if

$$p_z + s = \mathcal{O}(\eta), \tag{5.27a}$$

$$p_y + u = \mathcal{O}(\varepsilon), \quad p_x - v = \mathcal{O}(\varepsilon), \tag{5.27b}$$

$$u_x + v_y + \varepsilon w_z = \mathcal{O}(\varepsilon^2). \tag{5.27c}$$

If we replace $\mathcal{O}(\eta)$ and $\mathcal{O}(\varepsilon)$ by zero in (5.27a) and (5.27b), then the resulting equations are called the *hydrostatic assumption* and *geostrophic approximation*, respectively.

The second time derivatives are bounded independently of ε, if

$$\frac{d}{dt}(p_z + s) = \mathcal{O}(\eta), \tag{5.28a}$$

$$\frac{d}{dt}(p_y + u) = \mathcal{O}(\varepsilon), \quad \frac{d}{dt}(p_x - v) = \mathcal{O}(\varepsilon), \tag{5.28b}$$

$$\frac{d}{dt}(u_x + v_y + \varepsilon w_z) = \mathcal{O}(\varepsilon^2) \tag{5.28c}$$

(5.28) gives us

$$\begin{aligned} \varepsilon^2 \frac{d}{dt}(p_z + s) &= \varepsilon^2 \left(\frac{dp}{dt}\right)_z - \varepsilon^2 H_1 + \varepsilon^2 \frac{ds}{dt} \\ &= (u_x + v_y + \varepsilon w_z)_z + \varepsilon^2 w - \varepsilon^2 H_1 \\ &= \mathcal{O}(\varepsilon^2 \eta), \end{aligned} \tag{5.29}$$

where

$$H_1 = u_z p_x + v_z p_y + \varepsilon w_z p_z = u_z v - v_z u + \mathcal{O}(\varepsilon).$$

Thus (5.28a) gives us an improvement of (5.27c). If we replace the $\mathcal{O}(\varepsilon\eta)$ term by zero, then the resulting relation is called *Richardson's equation.*

Correspondingly, (5.28b) and (5.28c) lead to improvements in (5.27b) and (5.27c), respectively.

The primitive equations, that are often used in weather prediction models, are given by

$$\begin{aligned} \frac{ds}{dt} - w &= 0, \\ \varepsilon \frac{du}{dt} + p_x + u &= 0, \\ \varepsilon \frac{dv}{dt} + p_y - v &= 0, \\ p_z + s &= 0, \end{aligned}$$

$$(u_x + v_y + \varepsilon w_z)_z + \varepsilon^2 w - \varepsilon^2 H_1 = 0.$$

Their mathematical properties are discussed in Browning and Kreiss (1985) and Oliger and Sundström (1978).

Instead of pursuing the initialization to make more and more time derivatives bounded independently of ε it is easier to achieve this by iteration. We shall first derive a set of equations, which will determine the slow solution to order $\mathcal{O}(\varepsilon)$.

Ler $\xi = v_x - u_y$ denote the horizontal vorticity and use the notation

$$\frac{\mathrm{d_H}}{\mathrm{d}t} = \partial/\partial t + u\,\partial/\partial x + v\,\partial/\partial y.$$

(5.27) gives us the balance equation

$$\Delta_2 p = \xi + \mathcal{O}(\varepsilon), \quad \Delta_2 = \partial^2/\partial x^2 + \partial^2/\partial y^2. \tag{5.30}$$

Differentiating the horizontal momentum equations results in

$$\varepsilon\Big(\frac{\mathrm{d_H}\xi}{\mathrm{d}t} + \xi(u_x + v_y)\Big) + u_x + v_y = \mathcal{O}(\varepsilon^2).$$

Therefore, by (5.27c),

$$\frac{\mathrm{d_H}\xi}{\mathrm{d}t} = w_z + \mathcal{O}(\varepsilon). \tag{5.31}$$

The first equation of (5.26) tells us that

$$w_z = \frac{\mathrm{d}s_z}{\mathrm{d}t} + u_z s_x + v_z s_y + \mathcal{O}(\varepsilon),$$

i.e. by (5.27a) and (5.27b),

$$\begin{aligned} \frac{\mathrm{d_H}}{\mathrm{d}t}(\xi - s_x) &= u_z s_x + v_z s_y + \mathcal{O}(\varepsilon) \\ &= -u_z p_{xz} - v_z p_{yz} + \mathcal{O}(\varepsilon) = \mathcal{O}(\varepsilon). \end{aligned} \tag{5.32}$$

By (5.30) and (5.27c) we can also write (5.32) as

$$\frac{\mathrm{d_H}}{\mathrm{d}t}\Delta p = \mathcal{O}(\varepsilon).$$

Therefore the slow solution satisfies to a first approximation

$$\begin{aligned} \frac{\mathrm{d_H}}{\mathrm{d}t}\Delta p &= 0, \\ p_z + s &= 0, \quad p_y + u = 0, \quad p_x - v = 0, \\ w &= \frac{\mathrm{d_H}s}{\mathrm{d}t}. \end{aligned} \tag{5.33}$$

Higher order approximations are obtained by iteration.

Remark. If we had not made the simplification (5.25), then (5.33) would be slightly more complicated. It would still be a well posed problem.

We have assumed that we can apply our theory. We will now discuss this question. We can symmetrize the equations by introducing new variables

$$\sqrt{\varepsilon}\, p = \tilde{p}, \quad \sqrt{\eta}\, w = \tilde{w},$$

and obtain

$$\begin{aligned}
\frac{\mathrm{d}s}{\mathrm{d}t} - \frac{1}{\sqrt{\eta}}\tilde{w} &= 0, \\
\frac{\mathrm{d}\tilde{p}}{\mathrm{d}t} + \frac{1}{\varepsilon^{3/2}}(u_x + v_y) + \frac{1}{\sqrt{\varepsilon\eta}}\tilde{w}_z &= 0, \\
\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{1}{\varepsilon^{3/2}}\tilde{p}_x - \frac{1}{\varepsilon}v &= 0, \\
\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{1}{\varepsilon^{3/2}}\tilde{p}_y + \frac{1}{\varepsilon}u &= 0, \\
\frac{\mathrm{d}\tilde{w}}{\mathrm{d}t} + \frac{1}{\sqrt{\varepsilon\eta}}\tilde{p}_z + \frac{1}{\sqrt{\eta}}s &= 0,
\end{aligned} \tag{5.34}$$

where now

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \frac{\varepsilon}{\sqrt{\eta}}\tilde{w}\frac{\partial}{\partial z}. \tag{5.35}$$

Therefore our theory applies, provided

$$\frac{\varepsilon}{\sqrt{\eta}} \leq \text{ constant}, \tag{5.36}$$

otherwise the term $(\varepsilon/\sqrt{\eta})\tilde{w}\,\partial/\partial z$ becomes large. We shall assume that

$$\eta = \varepsilon^2. \tag{5.37}$$

In Browning *et al.* (1990) we have proved

Theorem 5.3 Assume that (5.26) with $\eta \gg \varepsilon^2$ has a solution with derivatives bounded independently of ε. We commit an error of order $\mathcal{O}(\varepsilon^2)$, if we change η to ε^2 and solve the new system with the same initial data.

Thus we can use the system (5.26) with $\eta = \varepsilon^2$ to obtain the desired slow solution up to terms of order $\mathcal{O}(\varepsilon^2)$, provided it exists. Again we can use Richardson extrapolation to approximate it to higher order.

The question, whether for $\eta \ll \varepsilon^2$ the system (5.26) has slow solutions, is not clear. Numerical calculations seem to indicate that it is so. However, if one linearizes (5.26) around a slow solution $U, V, \ldots$, then the linearized equations are unstable, if the sheer U_z, V_z is large compared with $\varepsilon/\sqrt{\eta}$, i.e. if we locally freeze the coefficients, then there are waves, which grow like $\exp(\alpha t)$, $\alpha = (|U_z| + |V_z|)\varepsilon/\sqrt{\eta}$. Further investigations are necessary.

There are other applications. For example, in Browning and Kreiss (1982) some problems in plasma physics are discussed, in Browning *et al.* (1980) and

Browning and Kreiss (1982) the shallow water equations are treated and in Raviart (1991) approximative models of Maxwell's equation are investigated.

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